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On the distribution of consecutive square-free numbers of the form $[\alpha \mathrm{n}], [\alpha \mathrm{n}] + 1$

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Abstract: In the present paper we show that there exist infinitely many consecutive square-free numbers of the form $[\alpha n]$, $[\alpha n] + 1$, where $\alpha > 1$ is irrational number with bounded partial quotient or irrational algebraic number.
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1 Notations

Let N be a sufficiently large positive integer. By ε we denote an arbitrary small positive number, not necessarily the same in different occurrences. We denote by $\mu(n)$ the Möbius function and by $\tau(n)$ the number of positive divisors of n. As usual $[t]$ and $\{t\}$ denote the integer part, respectively, the fractional part of t. Let $||t||$ be the distance from t to the nearest integer. Instead of $m \equiv n \pmod k$ we write for simplicity $m \equiv n\,(k)$. Moreover $e(t)$=exp($2\pi it$) and $\psi(t) = \{t\} - 1/2$. Let $\alpha > 1$ be irrational number with bounded partial quotient or irrational algebraic number.

Denote

$$\sigma = \prod_p \left(1 - \frac{2}{p^2}\right) . \tag{1}$$

We define the characteristic function $\omega_\alpha(x)$ in the interval $(0, 1]$ as follows

$$\omega_\alpha(x) = \begin{cases} 1\,, & \text{if } 1 - \frac{1}{\alpha} < x < 1\,; \\ \frac{1}{2}\,, & \text{if } x = 1 - \frac{1}{\alpha} \text{ or } x = 1\,; \\ 0\,, & \text{otherwise}\,; \end{cases} \tag{2}$$

and we extend it periodically to all real line.

2 Introduction and statement of the result

The problem for the consecutive square-free numbers arises in 1932 when Carlitz [3] proved that

$$\sum_{n\le N}\mu^2(n)\mu^2(n+1)=\sigma N+\mathcal{O}\big(N^{2/3+\varepsilon}\big)\,, \tag{3}$$

where σ is denoted by (1).

Subsequently in 1949 Mirsky [9] improved the error term of (3) to

$$\sum_{n\le N}\mu^2(n)\mu^2(n+1)=\sigma N+\mathcal{O}\left(N^{2/3}(\log N)^{4/3}\right)\,. \tag{4}$$

Further in 1984 Heath-Brown [7] improved the error term of (4) to

$$\sum_{n\le N}\mu^2(n)\mu^2(n+1)=\sigma N+\mathcal{O}\big(N^{7/11}(\log N)^{7}\big)\,. \tag{5}$$

Finally in 2014 Reuss [10] improved the error term of (5) to

$$\sum_{n\le N}\mu^2(n)\mu^2(n+1)=\sigma N+\mathcal{O}\big(N^{(26+\sqrt{433})/81+\varepsilon}\big) \tag{6}$$

and this is the best result up to now.

In 2008 Güloğlu and Nevans [6] showed that there exist infinitely many square-free numbers of the form $[\alpha n]$, where $\alpha>1$ is irrational number of finite type. More precisely they proved that the asymptotic formula

$$\sum_{n\le N}\mu^2([\alpha n])=\frac{6}{\pi^2}N+\mathcal{O}\left(\frac{N\log\log N}{\log N}\right)$$

holds.

On the other hand in 2009 Abercrombie and Banks[1] showed that for almost all $\alpha>1$ the asymptotic formula

$$\sum_{n\le N}\mu^2([\alpha n])=\frac{6}{\pi^2}N+\mathcal{O}\left(N^{\frac{2}{3}+\varepsilon}\right)$$

holds, however this result provides no specific value of α.

Subsequently in 2013 Victorovich [13] proved that when $\alpha>1$ is irrational number with bounded partial quotient or irrational algebraic number, then the asymptotic formula

$$\sum_{n\le N}\mu^2([\alpha n])=\frac{6}{\pi^2}N+\mathcal{O}\left(AN^{\frac{5}{6}}\log^5 N\right)$$

holds. Here $A=A(N)=\max\limits_{1\le m\le N^2}\tau(m)$.

In 2018 the author [4] showed that for any fixed $1<c<22/13$ there exist infinitely many consecutive square-free numbers of the form $[n^c]$, $[n^c]+1$.

Recently the author [5] proved that there exist infinitely many consecutive square-free numbers of the form x^2+y^2+1, x^2+y^2+2.

Define

$$S(N,\alpha)=\sum_{n\le N}\mu^2([\alpha n])\mu^2([\alpha n]+1)\,. \tag{7}$$

Motivated by these results and following the method of Victorovich [13] we shall prove the following theorem.

Theorem 1. *Let $\alpha>1$ be irrational number with bounded partial quotient or irrational algebraic number. Then for the sum $S(N,\alpha)$ defined by* (7) *the asymptotic formula*

$$S(N,\alpha)=\sigma N+\mathcal{O}\left(N^{\frac{5}{6}+\varepsilon}\right) \tag{8}$$

holds. Here σ is defined by (1).

3 Lemmas

Lemma 1. *For the function $\omega_\alpha(x)$ defined by* (2) *the formula*

$$\omega_\alpha(x)=\frac{1}{\alpha}+\psi(x)-\psi\left(x+\frac{1}{\alpha}\right)$$

holds.

Proof. See ([2], p. 480). □

Lemma 2. *For every $J\ge 2$, we have*

$$\psi(t)=\sum_{1\le|k|\le J}a(k)e(kt)+\mathcal{O}\left(\sum_{|k|\le J}b(k)e(kt)\right),\quad a(k)\ll 1/|k|\,,\quad b(k)\ll 1/J\,.$$

Proof. See [11]. □

Lemma 3. *If $X\ge 1$, then*

$$\Big|\sum_{n\le X}e(\alpha n)\Big|\le \min\left(X,\frac{1}{2||\alpha||}\right)\,.$$

Proof. See ([8], Ch. 6, §2). □

Lemma 4. *Suppose that $X,Y\ge 1,\ \lambda=\frac{a}{q}+\frac{\theta}{q^2},\ q\ge 1,\ (a,q)=1,\ |\theta|\le 1$. Then*

$$\sum_{n\le X}\min\left(Y,\frac{1}{||\lambda n||}\right)\ll\frac{XY}{q}+(X+q)\log 2q\,.$$

Proof. See ([12], Lemma 1). □

4 Proof of the Theorem

The equality $m = [\alpha n]$ is tantamount to $\alpha n - 1 < m < \alpha n$, $\frac{m}{\alpha} < n < \frac{m}{\alpha} + \frac{1}{\alpha}$, i.e. $\left\{\frac{m}{\alpha}\right\} > 1 - \frac{1}{\alpha}$. Then from (7) we get

$$S(N,\alpha) = \sum_{n\le N} \mu^2([\alpha n])\mu^2([\alpha n]+1) = \sum_{\substack{m\le\alpha N\\ \{\frac{m}{\alpha}\}>1-\frac{1}{\alpha}}} \mu^2(m)\mu^2(m+1)$$

$$= \sum_{m\le\alpha N} \mu^2(m)\mu^2(m+1)\omega_\alpha\left(\frac{m}{\alpha}\right). \tag{9}$$

Now (9) and Lemma 1 give us

$$S(N,\alpha) = \frac{1}{\alpha}\sum_{m\le\alpha N}\mu^2(m)\mu^2(m+1) + \sum_{m\le\alpha N}\mu^2(m)\mu^2(m+1)\left[\psi\left(\frac{m}{\alpha}\right) - \psi\left(\frac{m+1}{\alpha}\right)\right]$$

$$= \frac{1}{\alpha}S_1(N,\alpha) + S_2(N,\alpha), \tag{10}$$

where

$$S_1(N,\alpha) = \sum_{m\le\alpha N}\mu^2(m)\mu^2(m+1), \tag{11}$$

$$S_2(N,\alpha) = \sum_{m\le\alpha N}\mu^2(m)\mu^2(m+1)\left[\psi\left(\frac{m}{\alpha}\right) - \psi\left(\frac{m+1}{\alpha}\right)\right]. \tag{12}$$

Estimation of $\mathbf{S_1}(\mathbf{N},\alpha)$

Bearing in mind (6) and (11) we obtain

$$S_1(N,\alpha) = \sigma\alpha N + \mathcal{O}\big(N^{(26+\sqrt{433})/81+\varepsilon}\big), \tag{13}$$

where σ is denoted by (1).

Estimation of $\mathbf{S_2}(\mathbf{N},\alpha)$

Let

$$J = \sqrt{\alpha N}. \tag{14}$$

From (12) and Lemma 2 it follows

$$S_2(N,\alpha) = \sum_{m\le\alpha N}\mu^2(m)\mu^2(m+1)\sum_{1\le|k|\le J} a(k)e\left(\frac{km}{\alpha}\right)\left[1 - e\left(\frac{k}{\alpha}\right)\right]$$

$$+ \mathcal{O}\left(\sum_{m\le\alpha N}\mu^2(m)\mu^2(m+1)\sum_{|k|\le J} b(k)e\left(\frac{km}{\alpha}\right)\left[1 + e\left(\frac{k}{\alpha}\right)\right]\right)$$

$$= \sum_{1\le|k|\le J} a(k)\left[1 - e\left(\frac{k}{\alpha}\right)\right]\sum_{m\le\alpha N}\mu^2(m)\mu^2(m+1)e\left(\frac{km}{\alpha}\right)$$

$$+ \mathcal{O}\left(\sum_{|k|\le J} b(k)\left[1 + e\left(\frac{k}{\alpha}\right)\right]\sum_{m\le\alpha N}\mu^2(m)\mu^2(m+1)e\left(\frac{km}{\alpha}\right)\right)$$

$$= S_3(N,\alpha) + \mathcal{O}\Big(S_4(N,\alpha)\Big), \tag{15}$$

where

$$S_3(N,\alpha)=\sum_{1\le|k|\le J}a(k)\left[1-e\left(\frac{k}{\alpha}\right)\right]\sum_{m\le\alpha N}\mu^2(m)\mu^2(m+1)e\left(\frac{km}{\alpha}\right), \qquad (16)$$

$$S_4(N,\alpha)=\sum_{|k|\le J}b(k)\left[1+e\left(\frac{k}{\alpha}\right)\right]\sum_{m\le\alpha N}\mu^2(m)\mu^2(m+1)e\left(\frac{km}{\alpha}\right). \qquad (17)$$

Using (16) and Lemma 2 we find

$$S_3(N,\alpha)\ll\sum_{1\le|k|\le J}\frac{1}{k}\left|\sum_{m\le\alpha N}\mu^2(m)\mu^2(m+1)e\left(\frac{km}{\alpha}\right)\right|. \qquad (18)$$

From (14), (17) and Lemma 2 we get

$$S_4(N,\alpha)\ll\frac{1}{J}\sum_{1\le|k|\le J}\left|\sum_{m\le\alpha N}\mu^2(m)\mu^2(m+1)e\left(\frac{km}{\alpha}\right)\right|+\sqrt{N}. \qquad (19)$$

In order to estimate the sums $S_3(N,\alpha)$ and $S_4(N,\alpha)$ we shall prove the following lemma.

Lemma 5. *Let $\alpha>1$ be irrational number with bounded partial quotient or irrational algebraic number. Then for the sum*

$$\Sigma=\sum_{1\le k\le J}\frac{1}{k}\left|\sum_{m\le\alpha N}\mu^2(m)\mu^2(m+1)e(\lambda km)\right|. \qquad (20)$$

where $\lambda=\frac{1}{\alpha}$, the estimation

$$\Sigma\ll N^{\frac{5}{6}+\varepsilon},$$

holds.

Proof. Using (20) and the well-known identity $\mu^2(m)=\sum_{d^2|m}\mu(d)$ we write

$$\Sigma=\sum_{1\le k\le J}\frac{1}{k}\left|\sum_{m\le\alpha N}\left(\sum_{d^2|m}\mu(d)\right)\left(\sum_{t^2|m+1}\mu(t)\right)e(\lambda km)\right|$$

$$=\sum_{1\le k\le J}\frac{1}{k}\left|\sum_{d\le\sqrt{\alpha N}}\mu(d)\sum_{t\le\sqrt{\alpha N+1}}\mu(t)\sum_{\substack{r\le\frac{\alpha N}{d^2}\\ rd^2+1\equiv0\,(t^2)}}e(\lambda krd^2)\right|$$

$$\le\sum_{1\le k\le J}\frac{1}{k}\sum_{d\le\sqrt{\alpha N}}\sum_{t\le\sqrt{\alpha N+1}}\left|\sum_{\substack{r\le\frac{\alpha N}{d^2}\\ rd^2+1\equiv0\,(t^2)}}e(\lambda krd^2)\right|.$$

Splitting the range of k, d and t into dyadic subintervals we obtain

$$\Sigma \ll (\log N)^3 \max_{\substack{1\le K\le J/2 \\ 1\le D\le \sqrt{\alpha N}/2 \\ 1\le T\le \sqrt{\alpha N+1}/2}} \Sigma_0(K,D,T)\,, \tag{21}$$

where

$$\Sigma_0(K,D,T) = \sum_{K\le k\le 2K} \frac{1}{k} \sum_{D\le d\le 2D} \sum_{T\le t\le 2T} \left| \sum_{\substack{r\le \frac{\alpha N}{d^2} \\ rd^2+1\equiv 0\,(t^2)}} e(\lambda k r d^2) \right| . \tag{22}$$

If $(d,\, t) > 1$ then the sum $\Sigma_0(K,D,T)$ is empty. Suppose now that $(d,\, t) = 1$. Then the congruence $rd^2+1 \equiv 0\,(t^2)$ is equivalent to $r \equiv r_0\,(t^2)$, where r_0 is some integer with $1 \le r_0 \le t^2$. From the last consideration and (22) it follows

$$\Sigma_0(K,D,T) \le \sum_{K\le k\le 2K} \frac{1}{k} \sum_{D\le d\le 2D} \sum_{T\le t\le 2T} \left| \sum_{s\le \frac{\alpha N - r_0 d^2}{d^2 t^2}} e(\lambda k s d^2 t^2) \right| . \tag{23}$$

Consider two cases.

Case 1. $DT \le (\alpha N)^{1/6}$.

The inequality (23) and Lemma 3 give us

$$\begin{aligned}\Sigma_0(K,D,T) &\ll \sum_{K\le k\le 2K} \frac{1}{k} \sum_{D\le d\le 2D} \sum_{T\le t\le 2T} \min\left(\frac{\alpha N}{d^2t^2}, \frac{1}{||\lambda k d^2 t^2||}\right) \\ &\le \sum_{K\le k\le 2K} \sum_{D\le d\le 2D} \sum_{T\le t\le 2T} \min\left(\frac{\alpha N}{kd^2t^2}, \frac{1}{||\lambda k d^2 t^2||}\right) \\ &\le \sum_{K\le k\le 2K} \sum_{D\le d\le 2D} \sum_{T\le t\le 2T} \min\left(\frac{\alpha N}{KD^2T^2}, \frac{1}{||\lambda k d^2 t^2||}\right) .\end{aligned} \tag{24}$$

Replacing $m = kd^2t^2$ from (24) we get

$$\begin{aligned}\Sigma_0(K,D,T) &\le \sum_{KD^2T^2\le m\le 32KD^2T^2} \Bigg(\sum_{\substack{d^2|m \\ d\le 2D}} 1\Bigg)\Bigg(\sum_{\substack{t^2|m \\ t\le 2T}} 1\Bigg) \min\left(\frac{\alpha N}{KD^2T^2}, \frac{1}{||\lambda m||}\right) \\ &\le \sum_{KD^2T^2\le m\le 32KD^2T^2} \tau^2(m) \min\left(\frac{\alpha N}{KD^2T^2}, \frac{1}{||\lambda m||}\right) \\ &\ll N^{\varepsilon} \sum_{KD^2T^2\le m\le 32KD^2T^2} \min\left(\frac{\alpha N}{KD^2T^2}, \frac{1}{||\lambda m||}\right) .\end{aligned} \tag{25}$$

Since α $\left(\text{therefore } \lambda = \frac{1}{\alpha}\right)$ is irrational number with bounded partial quotient or irrational algebraic number then λ can be represented in the form $\lambda = \frac{a}{q} + \frac{\theta}{q^2}$, $(\alpha N)^{\frac{1}{2}-\varepsilon} \ll q \ll (\alpha N)^{\frac{1}{2}}$, $(a,q) = 1$, $|\theta| \le 1$. This follows for example from ([13], Ch.2, Lemma 1.5, Lemma 1.6). Bearing in mind these considerations, (14), (25), Lemma 4, the inequalities $DT \leq (\alpha N)^{1/6}$ and $K \leq J$ we find

$$\Sigma_0(K, D, T) \ll N^{\varepsilon} \left(\frac{\alpha N}{q} + KD^2T^2 + q \right) \log N \ll N^{\frac{5}{6}+\varepsilon} . \tag{26}$$

Case 2. $DT > (\alpha N)^{1/6}$.

Using (14), (23), the trivial estimate, the inequalities $DT > (\alpha N)^{1/6}$ and $K \leq J$ we obtain

$$\Sigma_0(K, D, T) \leq \sum_{K \leq k \leq 2K} \frac{1}{k} \sum_{D \leq d \leq 2D} \sum_{T \leq t \leq 2T} \frac{\alpha N}{d^2 t^2} \ll \frac{\alpha N}{DT} \log K \ll N^{\frac{5}{6}+\varepsilon} . \tag{27}$$

From (21), (26) and (27) it follows

$$\Sigma \ll N^{\frac{5}{6}+\varepsilon} .$$

The lemma is proved. □

On the one hand (18) and Lemma 5 give us

$$S_3(N, \alpha) \ll N^{\frac{5}{6}+\varepsilon} . \tag{28}$$

On the other hand (19) and Lemma 5 imply

$$S_4(N, \alpha) \ll \sum_{1 \leq |k| \leq J} \frac{1}{k} \left| \sum_{m \leq \alpha N} \mu^2(m)\mu^2(m+1) e\left(\frac{km}{\alpha} \right) \right| + \sqrt{N} \ll N^{\frac{5}{6}+\varepsilon} . \tag{29}$$

By (15), (28) and (29) we find

$$S_2(N, \alpha) \ll N^{\frac{5}{6}+\varepsilon} . \tag{30}$$

The end of the proof

Bearing in mind (10), (13) and (30) we obtain the asymptotic formula (8).

The theorem is proved.

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